

ON p -GENERIC SPLITTING VARIETIES FOR MILNOR K-SYMBOLS MOD p

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Abstract: the goal of this paper is to prove that the p -generic splitting varieties constructed from symmetric powers for two tuples (a_1, a_2, \dots, a_n) and (b_1, b_2, \dots, b_n) are birationally isomorphic whenever the Milnor K-symbols $\{a_1, a_2, \dots, a_n\}$ and $\{b_1, b_2, \dots, b_n\}$ are equal in $K_n^M(k)/p$.

1. INTRODUCTION

The Bloch-Kato conjecture relates the Milnor K-theory mod p of a field k with the étale cohomology of k with coefficients in the twists of μ_p . More precisely, it claims that for all prime $p \geq 2$ and all weight $n \geq 0$ there exists an isomorphism

$$K_n^M(k)/p \cong H_{\text{ét}}^n(k, \mu_p^n)$$

The special case of $p = 2$, known as Milnor conjecture, was proven by V. Voevodsky in [7] in 1993. In 2003 he continued to prove the general case in the preprint [8]. However his proof assumed the existence of a splitting variety with certain properties for a given Milnor K-symbol $\{a_1, a_2, \dots, a_n\}$. This gap was later filled when M. Rost provided the construction for such splitting varieties in [4, 3]. They are p -generic, meaning they are generic with respect to splitting fields having no finite extension of degree relatively prime to p . Alternately, V. Voevodsky suggested a more geometric way to construct these varieties that uses symmetric powers. We detail this latter construction in section 2.

Following these papers, there arose some questions about p -generic splitting varieties. My advisor Alexander S. Merkurjev asked whether the splitting varieties constructed from symmetric powers for two tuples (a_1, a_2, \dots, a_n) and (b_1, b_2, \dots, b_n) are birationally isomorphic whenever the Milnor K-symbols $\{a_1, a_2, \dots, a_n\}$ and $\{b_1, b_2, \dots, b_n\}$ are equal in $K_n^M(k)/p$. I would like to thank him for posing this question and helping to partially answer it in this paper.

In section 3 we use Chain P-equivalence Theorem by R. Elman and T. Y. Lam together with properties of quadratic forms to settle the case when $p = 2$. Section 4 proves that when $n = 2, 3$, symmetric power construction yields what are to be expected up to birational isomorphism. In section 5, we look to use Markus Rost's Chain Lemma and induction to solve the rest of our problem.

Date: June 12, 2009.

Advisor: Alexander S. Merkurjev.

2. SYMMETRIC POWERS

Throughout this paper, p is a prime and k is the base field of characteristic 0 containing the p -th roots of unity. We fix an n -tuple $(a_1, \dots, a_n), a_i \in k$ such that the symbol $\{a_1, \dots, a_n\}$ is nontrivial in the Milnor K -group $K_n^M(k)/p$. Associated to this symbol are the following notions.

Definition 2.1. A field extension L/k is called a splitting field for $\{a_1, \dots, a_n\}$ if $\{a_1, \dots, a_n\} = 0$ in $K_n^M(L)/p$.

Definition 2.2. A smooth variety X is called a splitting variety for $\{a_1, \dots, a_n\}$ if $\{a_1, \dots, a_n\} = 0$ in $K_n^M(k(X))/p$. In addition, it is called a generic splitting variety for $\{a_1, \dots, a_n\}$ if any splitting field L for $\{a_1, \dots, a_n\}$ has a point in X , i.e. there exists a morphism $\text{Spec}(L) \rightarrow X$ over k .

Generic splitting varieties are only known to exist for $n \leq 3$ when p is odd and for all n when p is even. However, if L'/L is a finite extension of degree prime to p and L' splits $\{a_1, \dots, a_n\}$ then L also splits $\{a_1, \dots, a_n\}$ (using transfer and norm maps). Therefore we can relax our last definition.

Definition 2.3. A smooth variety X is called a p -generic splitting variety for $\{a_1, \dots, a_n\}$ if X splits $\{a_1, \dots, a_n\}$ and for L splitting $\{a_1, \dots, a_n\}$ there exists an extension L'/L of degree prime to p and an L' -valued point $\text{Spec}(L') \rightarrow X$ over k .

We now describe a symmetric power construction that produces the aforementioned p -generic splitting varieties. It was suggested by V. Voevodsky and explained by S. Joukhovitski in [6, 2].

Let X be a smooth and geometrically irreducible quasi-projective variety. The symmetric group S_p acts on the product X^p and induces the quotient variety $S^p(X)$. This quotient is then geometrically irreducible and normal. Note that S_p acts freely on $X^p \setminus \Delta$ and $U := (Y^p \setminus \Delta)/S_p$ is an open set in $S^p(X)$, where Δ is the union of all diagonals in X^p .

For every normal and irreducible scheme Y the set of morphisms $\text{Hom}(Y, S^p(X))$ can be identified with the set of all effective cycles $Z \subset X \times Y$ such that each component of Z is finite surjective over Y and that the degree of Z over Y is p . In particular, the identity map $S^p(X) \xrightarrow{id} S^p(X)$ corresponds to the incidence cycle $Z \subset X \times S^p(X)$. In fact Z is a closed subscheme equal to the image of the closed embedding $X \times S^{p-1}(X) \hookrightarrow X \times S^p(X)$ mapping (x, y) to $(x, x + y)$. Compose this with projection onto the second factor and we get $p : X \times S^{p-1}(X) \rightarrow X \times S^p(X) \rightarrow S^p(X)$. It is finite surjective of degree p . Thus we get a diagram

$$\begin{array}{ccccccc}
 S^{p-1}(X) \times X & \longleftarrow & p^{-1}(U) & \longleftarrow & X^p \setminus \Delta & & \\
 \downarrow p & & \downarrow p|_{p^{-1}(U)} & & \swarrow |S_p & & \\
 S^p(X) & \longleftarrow & U & \longleftarrow & V & \longleftarrow & W
 \end{array}$$

We see both leftmost arrows are Galois étale coverings, $p|_{p^{-1}(U)}$ is a finite étale map of degree p , and U is smooth. Furthermore $p_*(\mathcal{O}_{X \times S^{p-1}(X)})$ is a coherent $\mathcal{O}_{S^p(X)}$ -algebra and the sheaf $\mathcal{A} := p_*(\mathcal{O}_{X \times S^{p-1}(X)}|_U)$ is a locally free \mathcal{O}_U -algebra of rank p . This latter sheaf corresponds to a vector bundle $V := \text{Spec}(S^\bullet \mathcal{A}^\vee)$ of rank p over U . Here \mathcal{A}^\vee denotes the dual of \mathcal{A} and S^\bullet denotes its symmetric algebra. There is a well defined norm map $\mathcal{A} \xrightarrow{N} \mathcal{O}_U$. Locally N is a homogeneous polynomial of degree p , that is, $N \in S^m(\mathcal{A}^\vee)$.

The p -generic splitting variety $X(a_1, \dots, a_n)$ is then constructed by induction. For $n = 2$ one may choose $X = X(a_1, a_2)$ in the preceding construction to be the Severi-Brauer variety $SB(A)$ associated with the cyclic algebra $A = (a_1, a_2, \zeta_p)_k$. Suppose we have constructed the smooth projective geometrically irreducible p -generic splitting variety $X(a_1, \dots, a_{n-1})$ for $\{a_1, \dots, a_{n-1}\}$ of dimension $p^{n-2} - 1$. Again let that be X and let $W \subset V$ be the hypersurface defined by the equation $N - a_n = 0$. By [6, 2.1] W is smooth over U (and hence smooth) and geometrically irreducible. By resolution of singularities we can embed W as an open subvariety of a new smooth, projective and geometrically irreducible variety X' . Together [6, 2.4] and [6, 2.4] show this X' is the p -generic splitting variety $X(a_1, \dots, a_n)$ for $\{a_1, \dots, a_n\}$ that we seek.

Among other properties, [6, 1.7] shows this p -generic splitting variety for $\{a_1, \dots, a_n\} \in K_n^M(L)/p$ is a geometrically irreducible smooth projective variety of dimension $p^{n-1} - 1$.

Remark 2.4. The inductive construction could in fact start with $n = 1$. We describe explicitly what happens at this stage. Take $X = X(a_1) = \text{Spec}(L)$ where $L = k(\sqrt[p]{a_1})$. If \bar{k} is the separable closure of k then $\bar{X} = X \times_{\text{Spec}(k)} \text{Spec}(\bar{k})$ has p points, call them $1, 2, \dots, p-1, p$. From there,

$$\begin{aligned} \bar{X}^p &= \{\text{points on the diagonals}\} \sqcup \{(n_1, n_2, \dots, n_p) \mid 1 \leq n_i \leq p \text{ and } n_i \neq n_j \text{ for all } i, j\} \\ S^p(\bar{X}) &= \bar{X}^p / S_p = \{\text{classes of points on the diagonals}\} \sqcup \{\overline{(1, 2, \dots, p)}\} \\ \bar{X}^p \setminus \Delta &= \{(n_1, n_2, \dots, n_p) \mid 1 \leq n_i \leq p \text{ and } n_i \neq n_j \text{ for all } i, j\} \\ (\bar{X}^p \setminus \Delta) / S_p &= \{\overline{(1, 2, \dots, p)}\} \end{aligned}$$

The above square thus looks like this,

$$\begin{array}{ccc} \bar{X} \times S^{p-1}(\bar{X}) & \xleftarrow{\quad} & p^{-1}(U) = \{(n, \overline{(2, 3, \dots, p)}), 1 \leq n \leq p\} \\ \downarrow p & & \downarrow p|_{p^{-1}(U)} \\ S^p(\bar{X}) & \xleftarrow{\quad} & U = \{\overline{(1, 2, \dots, p)}\} \end{array}$$

and over k it looks like this,

$$\begin{array}{ccc} X \times S^{p-1}(X) & \xleftarrow{\quad} & p^{-1}(U) \cong \text{Spec}(L) \\ \downarrow p & & \downarrow p|_{p^{-1}(U)} \\ S^p(X) & \xleftarrow{\quad} & U \cong \text{Spec}(k) \end{array}$$

We will use this in theorem 3.7 and theorem 4.1.

Remark 2.5. Since our problem only concerns with birational isomorphism, we can always replace our varieties with birationally isomorphic ones when it suits our purpose but does not change our result. Or we can consider what happens with the generic fiber. For example, in theorem 3.7 we consider the residue field of the generic fiber of the map p in our construction without mentioning V, W , and X' .

3. WHEN $p = 2$

When $p = 2$ we turn the p -generic splitting variety X for $\{a_1, \dots, a_n\}$ into the birationally isomorphic Pfister quadric associated to the Pfister form $\langle\langle a_1, \dots, a_n \rangle\rangle$. Once in quadric territory, we show this quadric defined by the a_i 's is birationally isomorphic to the quadric defined by the b_i 's through a sequence of birationally isomorphic quadrics, using Chain P-equivalence Theorem for Pfister forms.

For a quadratic form φ , let A_φ denote its symmetric matrix and $D_k(\varphi)$ denote its values in k . Also for an n -tuple (a_1, \dots, a_n) , let φ_n denote the n -fold Pfister form $\langle\langle a_1, \dots, a_n \rangle\rangle = \prod_{i=1}^n \langle 1, -a_i \rangle$. Furthermore we associate to φ_n the subform $\psi_n = \langle\langle a_1, \dots, a_{n-1} \rangle\rangle \perp \langle a_n \rangle$ and denote the quadric defined by ψ_n as $Z(\psi_n)$, known as Pfister quadric. Below are a few more definitions, a general reference for quadratic forms is [5].

Definition 3.1. Two quadratic forms φ and φ' are said to be equivalent, written $\varphi \cong \varphi'$, if there exists $C \in GL(k)$ such that $A_{\varphi'} = CA_\varphi C^t$.

Definition 3.2. Two Pfister forms $\varphi = \langle\langle a_1, \dots, a_n \rangle\rangle$ and $\varphi' = \langle\langle a'_1, \dots, a'_n \rangle\rangle$ are said to be simply P-equivalent if there exist two indices i, j such that $\langle\langle a_i, a_j \rangle\rangle \cong \langle\langle a'_i, a'_j \rangle\rangle$ and $a_k = a'_k$ for $k \neq i, j$. More generally, they are said to be chain P-equivalent, written $\varphi \cong \varphi'$, if there exists a sequence of Pfister forms $\varphi_0, \varphi_1, \dots, \varphi_{n-1}, \varphi_m$ such that $\varphi = \varphi_0, \varphi' = \varphi_m$ and φ_i is simply P-equivalent to $\varphi_{i+1}, 0 \leq i \leq m-1$.

Clearly $\varphi \cong \varphi'$ implies $\varphi \cong \varphi'$. The converse statement was proven by R. Elman and T. Y. Lam in [3], called Chain P-equivalence Theorem. We conveniently recall the statement here, for use in our proposition 3.10.

Theorem 3.3. (*Chain P-equivalence Theorem*) Let φ and φ' be n -fold Pfister forms. Then $\varphi \cong \varphi'$ if and only if $\varphi \cong \varphi'$.

Definition 3.4. Two quadratic forms φ and φ' are said to be birationally equivalent if their function fields $k(\varphi)$ and $k(\varphi')$ are isomorphic. This happens if and only if the quadrics they define are birationally isomorphic.

We begin with a lemma about equivalent Pfister forms and the matrix that connects them.

Lemma 3.5. If φ_{n-1} and $\varphi_n = \langle 1, -b \rangle \varphi_{n-1}$ are Pfister forms with matrices $A_{\varphi_{n-1}}$ and A_{φ_n} and $c = \varphi_n(x_1, \dots, x_{2n})$, then $\varphi_n \cong \langle c \rangle \varphi_n$ via a matrix $C_n \in GL_{2n}(k(x_i))$, that is $C_n A_{\varphi_n} C_n^t = c A_{\varphi_n}$, which satisfies 2 properties,

- (1) $C_n^{-1} = \frac{C_n}{c}$ (hence $(C_n^t)^{-1} = \frac{C_n^t}{c}$ as well)
- (2) first row and first column of C_n are $(x_1 \dots x_{2^n})$ and $A_{\varphi_n}(x_1 \dots x_{2^n})^t$

Proof. By induction on n . For $n = 1$ and $c = x_1^2 - ax_2^2$, $\varphi_1 \cong c\varphi_1$ via $C_1 = \begin{pmatrix} x_1 & x_2 \\ -ax_2 & -x_1 \end{pmatrix}$ satisfying (1) and (2).

Next, write $A_{\varphi_n} = \begin{pmatrix} A_{\varphi_{n-1}} & 0 \\ 0 & -bA_{\varphi_{n-1}} \end{pmatrix}$ then $c = \varphi_n(x_1, \dots, x_{2^n}) = xA_{\varphi_n}x^t = s - bt \in D_k(\varphi_n)$ where $s = \varphi_{n-1}(x_1, \dots, x_{2^{n-1}})$ and $t = \varphi_{n-1}(x_{2^{n-1}+1}, \dots, x_{2^n})$ are in $D_k(\varphi_{n-1})$. By induction, $\varphi_{n-1} \cong \langle s \rangle \varphi_{n-1}$ via a matrix $C \in GL_{2^{n-1}}(k(x_1, \dots, x_{2^{n-1}}))$, that is, $CA_{\varphi_{n-1}}C^t = sA_{\varphi_{n-1}}$, with

- (1) $C^{-1} = \frac{C}{s}$ (hence $(C^t)^{-1} = \frac{C^t}{s}$)
- (2) first row and first column of C are $(x_1 \dots x_{2^{n-1}})$ and $A_{\varphi_{n-1}}(x_1, \dots, x_{2^{n-1}})^t$

Similarly, $\varphi_{n-1} \cong \langle t \rangle \varphi_{n-1}$ via $C' \in GL_{2^{n-1}}(F(x_{2^{n-1}+1}, \dots, x_{2^n}))$ with the same properties. From this, we have

i) $\varphi_n \cong \langle s \rangle \varphi_{n-1} \perp \langle -b \rangle \langle t \rangle \varphi_{n-1} = \langle s, -bt \rangle \varphi_{n-1}$ with

$$\begin{pmatrix} C & 0 \\ 0 & C' \end{pmatrix} \begin{pmatrix} A_{\varphi_{n-1}} & 0 \\ 0 & -bA_{\varphi_{n-1}} \end{pmatrix} \begin{pmatrix} C^t & 0 \\ 0 & C'^t \end{pmatrix} = \begin{pmatrix} sA_{\varphi_{n-1}} & 0 \\ 0 & -btA_{\varphi_{n-1}} \end{pmatrix}$$

j) $\langle s, -bt \rangle \varphi_{n-1} \cong \langle c, -cbst \rangle \varphi_{n-1}$ with

$$\begin{pmatrix} I & I \\ btI & sI \end{pmatrix} \begin{pmatrix} sA_{\varphi_{n-1}} & 0 \\ 0 & -btA_{\varphi_{n-1}} \end{pmatrix} \begin{pmatrix} I & btI \\ I & sI \end{pmatrix} = \begin{pmatrix} cA_{\varphi_{n-1}} & 0 \\ 0 & -cbstA_{\varphi_{n-1}} \end{pmatrix}$$

k) Let $D = (CC')^{-1} = C'^{-1}C^{-1} = \frac{C'C}{ts}$, then $\langle c, -cbst \rangle \varphi_{n-1} \cong \langle c, -cb \rangle \varphi_{n-1} = \langle c \rangle \varphi_n$ with

$$\begin{aligned} & \begin{pmatrix} I & 0 \\ o & D \end{pmatrix} \begin{pmatrix} cA_{\varphi_{n-1}} & 0 \\ 0 & -cbstA_{\varphi_{n-1}} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & D^t \end{pmatrix} = \begin{pmatrix} cA_{\varphi_{n-1}} & 0 \\ 0 & -cbstDA_{\varphi_{n-1}} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & D^t \end{pmatrix} \\ & = \begin{pmatrix} cA_{\varphi_{n-1}} & 0 \\ 0 & -cbstDA_{\varphi_{n-1}}D^t \end{pmatrix} = \begin{pmatrix} cA_{\varphi_{n-1}} & 0 \\ 0 & -cbst\frac{A_{\varphi_{n-1}}}{st} \end{pmatrix} = \begin{pmatrix} cA_{\varphi_{n-1}} & 0 \\ 0 & -cbA_{\varphi_{n-1}} \end{pmatrix} \end{aligned}$$

l) Putting (i),(j), (k) together, $\varphi_n \cong \langle s \rangle \varphi_{n-1} \perp \langle -b \rangle \langle t \rangle \varphi_{n-1} = \langle s, -bt \rangle \varphi_{n-1} \cong \langle c, -cbst \rangle \varphi_{n-1} \cong \langle c, -cb \rangle \varphi_{n-1} = \langle c \rangle \varphi_n$ via C'_n where

$$\begin{aligned} C'_n &= \begin{pmatrix} I & 0 \\ o & D \end{pmatrix} \begin{pmatrix} I & I \\ btI & sI \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & C' \end{pmatrix} = \begin{pmatrix} I & I \\ btD & sD \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & C' \end{pmatrix} \\ &= \begin{pmatrix} C & C' \\ btC'^{-1}C^{-1}C & sC'^{-1}C^{-1}C' \end{pmatrix} = \begin{pmatrix} C & C' \\ bC' & \frac{C'CC'}{t} \end{pmatrix} \end{aligned}$$

At last, let $C_n = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} C'_n = \begin{pmatrix} C & C' \\ -bC' & -\frac{C'CC'}{t} \end{pmatrix}$ then its inverse $C_n^{-1} = \frac{C_n}{c}$, its first row and column are $(x_1 \dots x_{2^n})$ and $A_{\varphi_n}(x_1 \dots x_{2^n})^t$, and $C_n A_{\varphi_n} C_n^t = \begin{pmatrix} I & 0 \\ o & -I \end{pmatrix} C'_n A_{\varphi_n} C_n'^t \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = cA_{\varphi_n}$

The last equality can be verified directly,

$$\begin{aligned}
C_n A_{\varphi_n} C_n^t &= \begin{pmatrix} C & C' \\ -bC' & -\frac{C'CC'}{t} \end{pmatrix} \begin{pmatrix} A_{\varphi_{n-1}} & 0 \\ 0 & -bA_{\varphi_{n-1}} \end{pmatrix} \begin{pmatrix} C^t & -bC'^t \\ C'^t & -\frac{C'^t C^t C'^t}{t} \end{pmatrix} \\
&= \begin{pmatrix} CA_{\varphi_{n-1}} & -bC'A_{\varphi_{n-1}} \\ -bC'A_{\varphi_{n-1}} & \frac{b}{t}C'CC'A_{\varphi_{n-1}} \end{pmatrix} \begin{pmatrix} C^t & -bC'^t \\ C'^t & -\frac{C'^t C^t C'^t}{t} \end{pmatrix} \\
&= \begin{pmatrix} CA_{\varphi_{n-1}}C^t - bC'A_{\varphi_{n-1}}C'^t & -bCA_{\varphi_{n-1}}C'^t + \frac{b}{t}C'A_{\varphi_{n-1}}C'^t C^t C'^t \\ -bC'A_{\varphi_{n-1}}C^t + \frac{b}{t}C'CC'A_{\varphi_{n-1}}C'^t & b^2C'A_{\varphi_{n-1}}C'^t - \frac{b}{t^2}C'CC'A_{\varphi_{n-1}}C'^t C^t C'^t \end{pmatrix} \\
&= \begin{pmatrix} sA_{\varphi_{n-1}} - btA_{\varphi_{n-1}} & -bCA_{\varphi_{n-1}}C'^t + bA_{\varphi_{n-1}}C^t C'^t \\ -bC'A_{\varphi_{n-1}}C^t + bC'CA_{\varphi_{n-1}} & b^2tA_{\varphi_{n-1}} - bsA_{\varphi_{n-1}} \end{pmatrix} \\
&= \begin{pmatrix} cA_{\varphi_{n-1}} & -bCA_{\varphi_{n-1}}C'^t + \frac{b}{s}CA_{\varphi_{n-1}}C^t C'^t \\ -bC'A_{\varphi_{n-1}}C^t + \frac{b}{s}C'CC'A_{\varphi_{n-1}}C'^t & -bcA_{\varphi_{n-1}} \end{pmatrix} \\
&= \begin{pmatrix} cA_{\varphi_{n-1}} & -bCA_{\varphi_{n-1}}C'^t + bCA_{\varphi_{n-1}}C'^t \\ -bC'A_{\varphi_{n-1}}C^t + bC'A_{\varphi_{n-1}}C^t & -bcA_{\varphi_{n-1}} \end{pmatrix} \\
&= \begin{pmatrix} cA_{\varphi_{n-1}} & 0 \\ 0 & -bcA_{\varphi_{n-1}} \end{pmatrix} \\
&= cA_{\varphi_n}
\end{aligned}$$

□

This next lemma is needed to show the residue field in theorem 3.7 stays the same.

Lemma 3.6. *The $n \times n$ matrix $M = \begin{pmatrix} a_1b_1 & a_1b_2 & \cdot & \cdot & a_1b_n \\ a_2b_1 & a_2b_2 & \cdot & \cdot & a_2b_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_nb_1 & a_nb_2 & \cdot & \cdot & a_nb_n \end{pmatrix}$ has characteristic polynomial*

$$\det(xI - M) = x^{n-1}(x - a_1b_1 - a_2b_2 - \dots - a_nb_n).$$

Proof. We consider what M does to the basis,

$$\begin{aligned}
k^n &\xrightarrow{M} k^n \\
(1, 0, \dots, 0) &\mapsto b_1(a_1, \dots, a_n) \\
(0, 1, \dots, 0) &\mapsto b_2(a_1, \dots, a_n) \\
&\vdots \\
(0, 0, \dots, 1) &\mapsto b_n(a_1, \dots, a_n)
\end{aligned}$$

Thus M sends the vector (a_1, \dots, a_n) to $\alpha(a_1, \dots, a_n)$ where $\alpha = a_1b_1 + a_2b_2 + \dots + a_nb_n$. Letting $v_1 = (a_1, \dots, a_n)$, we choose a basis $\{v_1, \dots, v_n\}$ such that $\ker(M) = \langle v_2, \dots, v_n \rangle$ and again look at what M does,

$$\begin{aligned}
k^n &\xrightarrow{M} k^n \\
v_1 &\mapsto (\alpha, 0, \dots, 0) \\
v_2 &\mapsto (0, \dots, 0)
\end{aligned}$$

$$\begin{array}{c} \cdot \\ \cdot \\ v_n \mapsto (0, \dots, 0) \end{array}$$

In this new basis M has canonical form $\begin{pmatrix} \alpha & 0 & \cdot & 0 \\ 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 \end{pmatrix}$

and $\det(xI - M) = \text{char}_M(x) = x^n - \alpha x^{n-1} = x^{n-1}(x - a_1 b_1 - a_2 b_2 - \dots - a_n b_n)$. \square

We are now ready to turn generic p -splitting varieties into quadrics defined by subforms of Pfister forms.

Theorem 3.7. *Under symmetric power construction, the p -generic splitting variety $X(a_1, \dots, a_n)$ for $\{a_1, \dots, a_n\} \in K_n^M(k)/2$ is birationally isomorphic to the Pfister quadric $Z(\psi_n) \hookrightarrow \mathbb{P}_k^{2^n-1}$ defined by the subform $\psi_n = \langle \langle a_1, \dots, a_{n-1} \rangle \rangle \perp \langle -a_n \rangle$ of the Pfister form $\varphi_n = \langle \langle a_1, \dots, a_n \rangle \rangle$.*

Proof. By induction on n . First we verify the case $n = 2$. We begin symmetric power construction with $X(a_1) = \text{Spec}(L)$ where $L = k(\sqrt{a_1})$. As described in remark 2.4, we get

$$\begin{array}{ccc} \text{Spec}(L) \times \text{Spec}(L) & \xleftarrow{\quad} & p^{-1}(U) \cong \text{Spec}(L) \\ \downarrow p & & \downarrow \\ S^2(\text{Spec}(L)) & \xleftarrow{\quad} & U \cong \text{Spec}(k) \end{array}$$

Hence $X(a_1, a_2) = Z(N_{L/k} - a_2) = Z(x_1^2 - a_1 x_2^2 - a_2)$ the hypersurface defined by the equation $N_{L/k} - a_2 = x_1^2 - a_1 x_2^2 - a_2 = 0$. Projectivization then gives $X(a_1, a_2) = Z(x_1^2 - a_1 x_2^2 - a_2 x_3^2) = Z(\psi_2) \hookrightarrow \mathbb{P}_k^2$ as required.

Next, by induction $X(a_1, \dots, a_{n+1}) \approx Z(\psi_{n+1})$. Write $\psi = \psi_{n+1} = \varphi_n \perp \langle -a_{n+1} \rangle = \langle 1 \rangle \perp \varphi' \perp \langle -a_{n+1} \rangle \cong \langle 1, -a_{n+1} \rangle \perp \varphi'$ where φ' is the pure subform of φ . From construction we get

$$(X_{n+1} \times X_{n+1}) \setminus \Delta \longrightarrow ((X_{n+1} \times X_{n+1}) \setminus \Delta)/s_2 \longrightarrow Gr(2, \mathbb{A}_k^{2^n+1})$$

Let $U = \langle u, v \rangle = \langle (1, 0, x_2, \dots, x_{2^n}), (0, 1, y_2, \dots, y_{2^n}) \rangle$ be the generic plane in $\mathbb{A}_k^{2^n+1}$ and moreover let $\{u, v\}$ be a basis for U . Over this basis the restriction $\psi_{k(x_i, y_i)}|_U$ has matrix form $\begin{pmatrix} \psi(u) & b(u, v) \\ b(u, v) & \psi(v) \end{pmatrix}$, where

$$\begin{array}{c} U \times U \xrightarrow{\quad b \quad} k \\ (u', v') \mapsto \frac{1}{2}(\psi(u' + v') - \psi(u') - \psi(v')) \end{array}$$

is the symmetric bilinear form associated to $\psi_{k(x_i, y_i)}|_U$.

The generic fiber is then the point $(r, s) \in U$ such that

$$\psi(r, s) = \psi(u, u)r^2 + 2b(u, v)rs + \psi(v, v)s^2 = 0$$

with residue field

$$qf(k(x_i, y_i)[\frac{r}{s}]/(\psi(u, u)(\frac{r}{s})^2 + 2b(u, v)\frac{r}{s} + \psi(v, v)) = k(x_i, y_j)(\sqrt{-\theta})$$

where

$$\begin{aligned} \theta &= \psi(u)\psi(v) - b(u, v)^2 \\ &= (1 + \varphi'(x_2, \dots, x_{2^n}))(-a_{n+1} + \varphi'(y_2, \dots, y_{2^n})) - b(u, v)^2 \\ &= (\varphi(1, x_2, \dots, x_{2^n}))(-a_{n+1} + \varphi'(y_2, \dots, y_{2^n})) - b(u, v)^2 \\ &= (-a_{n+1})\varphi(1, x_2, \dots, x_{2^n}) + \varphi(1, x_2, \dots, x_{2^n})\varphi(0, y_2, \dots, y_{2^n}) - b(u, v)^2 \end{aligned}$$

If we write $\varphi = \langle 1, c_2, \dots, c_{2^n} \rangle$ then by lemma 3.5, there exists a matrix

$$C_n = \begin{pmatrix} 1 & x_2 & . & . & x_{2^n} \\ c_2 x_2 & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ c_{2^n} x_{2^n} & . & . & . & . \end{pmatrix}$$

such that $\varphi(1, x_2, \dots, x_{2^n})\varphi(0, y_2, \dots, y_{2^n}) = \varphi((0, y_2, \dots, y_{2^n})C_n)$. So

$$\begin{aligned} \theta &= (-a_{n+1})\varphi(1, x_2, \dots, x_{2^n}) + \varphi((0, y_2, \dots, y_{2^n})C_n) - b(u, v)^2 \\ &= (-a_{n+1})\varphi(1, x_2, \dots, x_{2^n}) + \varphi((0, y_2, \dots, y_{2^n})A_\varphi(1, x_2, \dots, x_{2^n})^t, z_2, \dots, z_{2^n}) \\ &\quad - ((y_2, \dots, y_{2^n})A_{\varphi'}(x_2, \dots, x_{2^n})^t)^2 \\ &= (-a_{n+1})\varphi(1, x_2, \dots, x_{2^n}) + \varphi((y_2, \dots, y_{2^n})A_{\varphi'}(x_2, \dots, x_{2^n})^t, z_2, \dots, z_{2^n}) \\ &\quad - ((y_2, \dots, y_{2^n})A_{\varphi'}(x_2, \dots, x_{2^n})^t)^2 \\ &= (-a_{n+1})\varphi(1, x_2, \dots, x_{2^n}) + \varphi'(z_2, \dots, z_{2^n}) \end{aligned}$$

Above, we let $(z_1, z_2, \dots, z_{2^n}) = (0, y_2, \dots, y_{2^n})C_n$, which means $(z_2, \dots, z_{2^n}) = (y_2, \dots, y_{2^n})M$ where M is C_n without its first row and column. Since $C_n^2 = \varphi(1, x_2, \dots, x_{2^n})I$, it follows $M^2 = \varphi(1, x_2, \dots, x_{2^n})I - (c_i x_i x_j), 2 \leq i, j \leq 2^n$. By lemma 3.6, $\det(M^2) = \varphi(1, x_2, \dots, x_{2^n})^{2^n-2}$, $\det(M) = \varphi(1, x_2, \dots, x_{2^n})^{2^{n-1}-1}$, and $M \in GL_{2^n-1}(F(x_2, \dots, x_{2^n}))$. So the residue field stays the same

$$F(x_i, y_j)(\sqrt{-\theta}) = F(x_i, z_j)(\sqrt{-\theta})$$

It has quadratic norm

$$\begin{aligned} N(m + n\sqrt{-\theta}) &= m^2 - a_{n+1}\varphi(1, x_2, \dots, x_{2^n})n^2 + \varphi'(z_2, \dots, z_{2^n})n^2 \\ &= \varphi(m, nz_2, \dots, nz_{2^n}) - a_{n+1}\varphi(n, nx_2, \dots, nx_{2^n}) \\ &= \langle 1, -a_{n+1} \rangle \varphi(m, nz_2, \dots, nz_{2^n}, n, nx_2, \dots, nx_{2^n}) \\ &= \varphi_{n+1}(t_1, \dots, t_{2^{n+1}}) \end{aligned}$$

Therefore our projectivized $X(a_1, \dots, a_{n+2}) = Z(N - a_{n+2}t_{2^{n+1}+1}^2)$ is birationally isomorphic to $Z(\varphi_{n+1} \perp \langle -a_{n+2} \rangle) = Z(\psi_{n+2}) \hookrightarrow \mathbb{P}_k^{2^{n+1}}$ as wanted. \square

Next, we show that interchanging a_i and a_j or multiplying a_i by any nonzero $N_{k(\sqrt{a_j})/k}(u)$ in the symbol $\{a_1, \dots, a_n\}$ does not change its p -generic splitting variety. For this, we need two more lemmas about Pfister neighbors, the first one we will use toward our problem and the second one we will use toward our example 3.14.

Lemma 3.8. *If $\varphi = \langle\langle a_1, \dots, a_n \rangle\rangle$ is a anisotropic Pfister form then the two forms $\varphi \perp \langle -b\varphi \rangle \perp \langle -c \rangle$ and $\varphi \perp \langle -c\varphi \rangle \perp \langle -b \rangle$ are birationally equivalent.*

Proof. We connect the quadrics defined by these two forms by a sequence of birational ones. Let (x, y, z) be the generic zero for the form $\varphi \perp \langle -b\varphi \rangle \perp \langle -c \rangle$, then

$$\varphi(x) - b\varphi(y) - cz^2 = 0$$

Since φ is Pfister and $\varphi(y) \in D_{k(y)}(\varphi)$, it follows $\varphi \cong \varphi(y)\varphi$ over $k(y)$. That means there exists $C \in GL(k(y))$ such that $\varphi(x) = \varphi(y)\varphi(Cx)$. Let $x' = Cx$ then $k(x, y, z) = k(x', y, z)$ and

$$\varphi(y)\varphi(x') - b\varphi(y) - cz^2 = 0, \text{ hence}$$

$$\varphi(x') - b - c\frac{z^2}{\varphi(y)} = 0$$

Now let $y' = \frac{y}{\varphi(y)}$ then $k(x, y, z) = k(x', y', z)$ and

$$\varphi(x') - b - cz^2\varphi(y') = 0, \text{ hence}$$

$$\frac{\varphi(x')}{z^2} - \frac{b}{z^2} - c\varphi(y') = 0$$

At last let $x'' = \frac{x'}{z}, z' = \frac{1}{z}$ then (x'', y', z') is a generic zero for $\varphi \perp \langle -c\varphi \rangle \perp \langle -b \rangle$, $k(x, y, z) = k(x'', y', z')$ and

$$\varphi(x'') - c\varphi(y') - bz'^2 = 0$$

Hence the two forms $\varphi \perp \langle -b\varphi \rangle \perp \langle -c \rangle$ and $\varphi \perp \langle -c\varphi \rangle \perp \langle -b \rangle$ are birationally equivalent. \square

Lemma 3.9. *If $\varphi = \langle\langle a_1, \dots, a_n \rangle\rangle$ is a anisotropic Pfister form then the two forms $\varphi \perp \langle -b \rangle$ and $\varphi \perp \langle -b\varphi(x_0) \rangle$ with $\varphi(x_0) \neq 0$ are birationally equivalent. In particular $\varphi \perp \langle -b \rangle \approx \varphi \perp \langle -bN_{k(\sqrt{a_i})/k}(u) \rangle$ for $N_{k(\sqrt{a_i})/k}(u) \neq 0$.*

Proof. We use the same approach as in lemma 3.8. Let (x, y) be a generic zero for the form $\varphi \perp \langle -b\varphi(x_0) \rangle$, then

$$\varphi(x) - b\varphi(x_0)y^2 = 0, \text{ hence}$$

$$\varphi(x_0)\varphi(x) - b\varphi(x_0)^2y^2 = 0$$

Again $\varphi \cong \varphi(x_0)\varphi$ over k , i.e. there exists $C \in GL(k)$ such that $\varphi(Cx) = \varphi(x_0)\varphi(x)$. Let $x' = Cx, y' = \varphi(x_0)y$ then (x', y') is a generic zero for $\varphi \perp \langle -b \rangle$, $k(x, y) = k(x', y')$, and

$$\varphi(x') - by'^2 = 0$$

Therefore the two forms $\varphi \perp \langle -b \rangle$ and $\varphi \perp \langle -b\varphi(x_0) \rangle$ with $\varphi(x_0) \neq 0$ are birationally equivalent. The last statement follows when we choose x_0 such that $\varphi(x_0) = N_{k(\sqrt{a_i})/k}(u)$. \square

Proposition 3.10. *If two Pfister form φ and φ' are equivalent then their associated subforms ψ and ψ' are birationally equivalent.*

Proof. By Chain P-equivalence Theorem, $\varphi \cong \varphi'$. So there exists a sequence of Pfister forms $\varphi_0, \varphi_1, \dots, \varphi_t, \dots, \varphi_{m-1}, \varphi_m$ such that $\varphi = \varphi_0, \varphi' = \varphi_m$ and φ_t is simply P-equivalent to $\varphi_{t+1}, 0 \leq t \leq m-1$. Write $\varphi_t = \langle\langle a_1, \dots, a_i, \dots, a_j, \dots, a_n \rangle\rangle$ and $\varphi_{t+1} = \langle\langle a_1, \dots, a'_i, \dots, a'_j, \dots, a_n \rangle\rangle$ where $\langle\langle a_i, a_j \rangle\rangle \cong \langle\langle a'_i, a'_j \rangle\rangle$. If $i = j$ then there is nothing to do. Else we consider all the cases,

i) if $j \neq n$ then

$$\begin{aligned} \psi_t &= \langle\langle a_1, \dots, a_i, \dots, a_j, \dots, a_{n-1} \rangle\rangle \perp \langle -a_n \rangle \\ &\cong \langle\langle a_1, \dots, a'_i, \dots, a'_j, \dots, a_{n-1} \rangle\rangle \perp \langle -a_n \rangle \\ &= \psi_{t+1} \end{aligned}$$

j) if $i \neq n-1$ and $j = n$ then by lemma 3.8,

$$\begin{aligned} \psi_t &= \langle\langle a_1, \dots, a_i, \dots, a_{n-1} \rangle\rangle \perp \langle -a_j \rangle \\ &\approx \langle\langle a_1, \dots, a_i, \dots, a_j \rangle\rangle \perp \langle -a_{n-1} \rangle \\ &\cong \langle\langle a_1, \dots, a'_i, \dots, a'_j \rangle\rangle \perp \langle -a_{n-1} \rangle \\ &\approx \langle\langle a_1, \dots, a'_i, \dots, a_{n-1} \rangle\rangle \perp \langle -a'_j \rangle \\ &= \psi_{t+1} \end{aligned}$$

iii) if $i = n-1$ and $j = n$ then again by lemma 3.8,

$$\begin{aligned} \psi_t &= \langle\langle a_1, \dots, a_{n-2}, a_i \rangle\rangle \perp \langle -a_j \rangle \\ &\cong \langle\langle a_1, \dots, a_i, a_{n-2} \rangle\rangle \perp \langle -a_j \rangle \\ &\approx \langle\langle a_1, \dots, a_i, a_j \rangle\rangle \perp \langle -a_{n-2} \rangle \\ &\cong \langle\langle a_1, \dots, a'_i, a'_j \rangle\rangle \perp \langle -a_{n-2} \rangle \\ &\approx \langle\langle a_1, \dots, a'_i, a_{n-2} \rangle\rangle \perp \langle -a'_j \rangle \\ &\cong \langle\langle a_1, \dots, a_{n-2}, a'_i \rangle\rangle \perp \langle -a'_j \rangle \\ &= \psi_{t+1} \end{aligned}$$

In any case, $\psi_t \approx \psi_{t+1}$ for all i . Hence $\psi \approx \psi'$. \square

Remark 3.11. Let φ be a Pfister form of $\dim \geq 2$, $c \in k^*$, and φ_1 a nonzero subform of φ . Hamza Ahmad called (φ, c, φ_1) a Pfister triple, $\varphi \perp \langle c \rangle$ the base form, $\varphi \perp c\varphi_1$ the form defined by the triple, $\varphi \perp c\varphi$ the associated Pfister form, and any form similar to such a $\varphi \perp c\varphi_1$ a special Pfister neighbor. In this setting, the forms in lemma 3.8 and the forms in lemma 3.9 are pairwise special Pfister neighbors of the same dimensions and have the same associated Pfister forms $\varphi \otimes \langle\langle b, c \rangle\rangle$ and $\varphi \otimes \langle\langle b \rangle\rangle$, respectively. The lemmas then follow from his more general [1, 1.6].

Remark 3.12. One sees that lemmas 3.8 and 3.9 hold for any strongly multiplicative form φ as defined in [5]. The work lies with anisotropic Pfister forms. The remaining strongly multiplicative forms are isotropic, hence their function fields are rational and both lemmas become trivial.

Theorem 3.13. *The p -generic splitting varieties constructed for $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ are birationally isomorphic if $\{a_1, \dots, a_n\} = \{b_1, \dots, b_n\}$ in $K_n^M(k)/2$.*

Proof. By [2, 6.20], the two Pfister forms $\varphi = \langle\langle a_1, \dots, a_n \rangle\rangle$ and $\varphi' = \langle\langle b_1, \dots, b_n \rangle\rangle$ are equivalent. Proposition 3.10 now implies their associated subforms ψ and ψ' are birationally equivalent. By theorem 3.7, $X(a_1, \dots, a_n)$ and $X(b_1, \dots, b_n)$ are birationally isomorphic. \square

Example 3.14. For any nonzero $N_{k(\sqrt{a_i})/k}(u)$, we know $\{a_1, \dots, a_i, \dots, a_j, \dots, a_n\} = \{a_1, \dots, a_i, \dots, a_j N_{k(\sqrt{a_i})/k}(u), \dots, a_n\} \in K_n^M(k)/2$. By theorem 3.13, they share the same p -generic splitting variety. Or we can use theorem 3.7 and lemma 3.9, bypassing Chain P -equivalence Theorem to see this as well.

4. WHEN $n = 2, 3$

We show that when $n = 2, 3$, the p -generic splitting varieties produced by symmetric power construction are what we would expect up to birational isomorphism. When $n = 2$, the obvious guess is Severi-Brauer varieties associated to cyclic algebras.

Theorem 4.1. *The p -generic splitting variety $X(a, b) = Z(N - b)$ constructed for $\{a, b\} \in K_2^M(k)/p$ is birationally isomorphic to the Severi-Brauer variety $SB(A)$ associated to the cyclic algebra $A = (a, b, \zeta_p)_k$.*

Proof. Again if we start symmetric power construction with $X(a) = \text{Spec}(L)$ where $L = k(\sqrt[p]{a})$ then by remark 2.4 $X(a, b) = Z(N_{L/k} - b)$. We consider what happens in the split case, i.e. over a field F that splits $\{a, b\}$. Choose L as F then $A_L \cong M_p(L)$ and $SB(A_L) \cong \mathbb{P}_L^{p-1}$. Furthermore, if $G = \text{Gal}(L/k) \cong \langle \sigma \rangle$ the cyclic group of order p then over L the norm $N(u)$ splits in to a product $\prod_{i=0}^{p-1} \sigma^i(u)$ for every $u \in L$. Define

$$U_L = \{I \subset M_p(L) \mid I = \begin{pmatrix} \alpha_1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \alpha_p & \cdot & \cdot \end{pmatrix} M \mid \alpha_i \neq 0 \text{ for all } i \text{ and } M \in M_p(L)\}$$

then U_L is an open set in $SB(A_L)$ and we have a diagram

$$\begin{array}{ccccc} Z(N - b)_L & \xrightarrow{f_L} & U_L & \xrightarrow{\text{open}} & SB(A_L) \\ \downarrow /G & & \downarrow /G & & \downarrow /G \\ Z(N - b) & \xrightarrow{f} & U & \xrightarrow{\text{open}} & SB(A) \end{array}$$

If we abuse notation and write points in $SB(A_L)$ in projective coordinates then f_L can be described as follows,

$$\begin{array}{c} Z(N - b)_L \xrightarrow{f_L} U_L \\ (x, \sigma(x), \dots, \sigma^{p-1}(x)) \mapsto (x : x\sigma(x) : \dots : x\sigma(x)\dots\sigma^{p-1}(x)) \end{array}$$

We verify that f_L is G -equivariant,

$$\begin{aligned} f_L(\sigma \cdot (x, \sigma(x), \dots, \sigma^{p-1}(x))) &= f(\sigma(x), \sigma^2(x), \dots, x) \\ &= (\sigma(x) : \sigma(x)\sigma^2(x) : \dots : \sigma(x)\sigma^2(x)\dots\sigma^p(x)) \\ &= (\sigma(x) : \sigma(x)\sigma^2(x) : \dots : b) \end{aligned}$$

while

$$\begin{aligned} \sigma \cdot f_L(x, \sigma(x), \dots, \sigma^{p-1}(x)) &= \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & 1 & \cdot \\ b & \cdot & \cdot & \cdot & 1 \end{pmatrix} \begin{pmatrix} x \\ x\sigma(x) \\ \cdot \\ \cdot \\ x\sigma(x)\dots\sigma^{p-1}(x) \end{pmatrix} \\ &= (x\sigma(x) : x\sigma(x)\sigma^2(x) : \dots : bx) \\ &= (\sigma(x) : \sigma(x)\sigma^2(x) : \dots : b) \end{aligned}$$

In function fields, we have an isomorphism f_L ,

$$\begin{aligned} L(U_L) &= L(\frac{t_i}{t_0}) \xrightarrow{f_L} L(Z(N-b)_L) = L(x, \sigma(x), \dots, \sigma^{p-1}(x)) \\ \frac{t_i}{t_0} &\mapsto x\sigma(x)\dots\sigma^{i-1}(x) \quad \text{where } i = 0, \dots, p-1, \frac{t_p}{t_0} = b \end{aligned}$$

with inverse

$$\begin{aligned} L(x, \sigma(x), \dots, \sigma^{p-1}(x)) &\xrightarrow{f_L^{-1}} L(\frac{t_i}{t_0}) \\ \sigma^i(x) &\mapsto \frac{t_{i+1}}{t_i} \end{aligned}$$

We verify that f_L respects G -action,

$$f_L(\sigma \cdot \frac{t_i}{t_0}) = f_L(\frac{t_{i+1}}{t_1}) = f_L((\frac{t_{i+1}}{t_0})(\frac{t_1}{t_0})^{-1}) = x\sigma(x)\dots\sigma^i(x)(x)^{-1} = \sigma(x)\dots\sigma^i(x)$$

while

$$\sigma \cdot f_L(\frac{t_i}{t_0}) = \sigma \cdot (x\sigma(x)\dots\sigma^{i-1}(x)) = \sigma(x)\dots\sigma^i(x)$$

Therefore $Z(N-b)_L$ is birationally isomorphic to U_L . So $Z(N-b)$ is birationally isomorphic to U , hence to $SB(A)$. \square

This enables us to solve our problem in the case $n = 2$,

Corollary 4.2. *The p -generic splitting varieties constructed for $\{a_1, a_2\}$ and $\{b_1, b_2\}$ are birationally isomorphic if $\{a_1, a_2\} = \{b_1, b_2\}$ in $K_2^M(k)/p$.*

Proof. By the norm residue homomorphism, the classes of $(a_1, a_2, \zeta_p)_k$ and $(b_1, b_2, \zeta_p)_k$ are equal in the group $Br_p(k)$ of elements in the Brauer group $Br(k)$ whose p^{th} power is 1. Since they have the same dimension, $(a_1, a_2, \zeta_p)_k$ and $(b_1, b_2, \zeta_p)_k$ are isomorphic as algebras. Hence $SB((a_1, a_2, \zeta_p)_k) \cong SB((b_1, b_2, \zeta_p)_k)$. It follows from the theorem that $X(a_1, a_2) \approx X(b_1, b_2)$. \square

When $n = 3$, the candidate is reduced norm varieties.

Theorem 4.3. *The splitting variety $X(a, b, c) = Z(N-c)$ constructed for $\{a, b, c\} \in K_3^M(k)/p$ is birationally isomorphic to $Z(Nrd_{A/k} - c)$, where $A = (a, b, \zeta_p)_k$.*

Proof. Again we consider the split case. Let $L = k(\sqrt[p]{a})$ and use $SB(A)$ where $A = (a, b, \zeta_p)_k$ as the p -generic splitting variety for $\{a, b\}$. Then $A_L \cong M_p(L)$ and $SB(A_L) \cong \mathbb{P}_L^{p-1}$.

Our symmetric power construction looks like the front square over k and the back square over L ,

$$\begin{array}{ccccc}
 & & SB(A_L) \times S^{p-1}(SB(A_L)) & \xleftarrow{\quad} & p^{-1}(U_L) \\
 & \swarrow /G & \downarrow & \swarrow /G & \downarrow \\
 SB(A) \times SB^{p-1}(A) & \xleftarrow{\quad} & p^{-1}(U) & & \\
 \downarrow p & & \downarrow p_L & & \downarrow p_L \\
 & \swarrow /G & S^p(SB(A_L)) & \xleftarrow{\quad} & U_L \xleftarrow{\pi_L} V_L \\
 & \downarrow & \downarrow & \swarrow /G & \downarrow \\
 S^p(SB(A)) & \xleftarrow{\quad} & U & \xleftarrow{\pi} & V
 \end{array}$$

Now let X_L denote the variety of all étale subalgebras of degree p in $End_L(L^p)$. If each subalgebra $D \in X_L$ is generated by a matrix λ where $\lambda = (\lambda_1, \dots, \lambda_p)$ its diagonal form then we see S_p acts trivially on X_L by permuting the diagonal entries. So we have an S_p -equivariant map

$$\begin{aligned}
 U_L &\xrightarrow{f_L} X_L \\
 (u_1, \dots, u_p) &\mapsto D
 \end{aligned}$$

with inverse $f_L^{-1} : D \mapsto (u_1, \dots, u_p)$ where D is the étale subalgebra whose eigenspaces are the lines u_1, \dots, u_p . This map fits into the following commutative diagram,

$$\begin{array}{ccc}
 U_L & \xrightarrow{f_L} & X_L \\
 \downarrow /G & & \downarrow /G \\
 U & \xrightarrow{f} & X
 \end{array}$$

and we get vector bundles over the last diagram,

$$\begin{array}{ccccc}
 V_L & \xrightarrow{f_L^*} & V_{X_L} & & \\
 \pi_L \downarrow & \searrow & \downarrow \pi_{X_L} & \searrow & \\
 & V & \xrightarrow{f^*} & V_X & \\
 & \downarrow \pi & \downarrow & & \\
 U_L & \xrightarrow{f_L} & X_L & & \pi_X \downarrow \\
 & \searrow & \searrow & & \\
 & U & \xrightarrow{f} & X &
 \end{array}$$

If (u_1, \dots, u_p) is a point in U_L then $p_L^{-1}((u_1, \dots, u_p))$ consists of p points y_1, \dots, y_p where each y_i is of the form $(u_i, [u_1, \dots, \check{u}_i, \dots, u_p])$. From there, $\pi_L^{-1}((u_1, \dots, u_p)) = \{((u_1, \dots, u_p), x_1, \dots, x_p) \mid x_i \in L(y_i)\}$. Correspondingly, $\pi_{X_L}^{-1}(D) = \{(D, d) \mid d \in D\}$. Both are algebras of rank p over L . We can describe the back face of the cube pointwise,

$$\begin{array}{ccc}
 ((u_1, \dots, u_p), x_1, \dots, x_p) & \xrightarrow{f_L^*} & (D, \begin{pmatrix} x_1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot \\ \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & x_p \end{pmatrix}) \\
 \pi_L \downarrow & & \downarrow \pi_{X_L} \\
 (u_1, \dots, u_p) & \xrightarrow{f_L} & D
 \end{array}$$

Note that if $p(t) = a_1 t + \dots + a_p t^p$ and $D \ni d = p(\lambda)$ with eigenvalues $p(\lambda_i)$ then $f_L^{*-1}(D, d) = ((u_1, \dots, u_p), p(\lambda_1), \dots, p(\lambda_p))$.

Therefore in V_L and V_{X_L} we have two birational subvarieties $Z(N - c)_L$ and $Z(Nrd_{A_L/L} - c)$, since

$$\begin{aligned}
Z(N - c)_L &= \{(u_1, \dots, u_p), x_1, \dots, x_p \mid x_1 \dots x_p = c\} \\
&\cong \{(D, d) \mid D \subset A_L \text{ étale of rank } p \text{ and } d \in D \text{ with } N_{D/L}(d) = c\} \\
&= \{(D, d) \mid D \subset A_L \text{ étale of rank } p \text{ and } d \in D \text{ with } Nrd_{A_L/L}(d) = c\} \\
&\cong \{d \in A_L \mid \langle d \rangle \subset A_L \text{ étale of rank } p \text{ and } Nrd_{A_L/L}(d) = c\} \text{ via } (D, d) \mapsto d \\
&= \{d \in A_L \mid Nrd_{A_L/L}(d) = c\} \cap \{d \in A_L \mid \text{its minimal polynomial } m_d(t) \\
&\quad \text{is of degree } p\} \\
&= \{d \in A_L \mid Nrd_{A_L/L}(d) = c\} \cap \{d \in A_L \mid x_i \neq x_j \text{ for all of its} \\
&\quad \text{eigenvalues } x_i, x_j\} \\
&\approx \{d \in A_L \mid Nrd_{A_L/L}(d) = c\} \\
&= Z(Nrd_{A_L/L} - c)
\end{aligned}$$

Note that the intersection above is nonempty, it contains for example the diagonal matrix $(\frac{c}{\zeta_p^{(p-1)/2}}, \zeta_p, \dots, \zeta_p^{p-1})$, and the second set is open. Therefore our p -generic splitting variety $X(a, b, c) = Z(N - c)$ is birationally isomorphic to $Z(Nrd_{A/k} - c)$ over k . \square

Knowing that $X(a, b, c) = Z(N - c)$ is birational to $Z(Nrd_{A/k} - c)$, where $A = (a, b, \zeta_p)_k$ is an advantage. However, it is unclear whether $\{a, b, c\} = \{a', b', c'\} \in K_3^M(k)/p$ implies $Z(Nrd_{A/k} - c) \approx Z(Nrd_{A'/k} - c')$, where $A = (a, b, \zeta_p)_k$ and $A' = (a', b', \zeta_p)_k$. This stops us short of being able to draw the same conclusion for $n = 3$ as for $n = 2$.

5. WHEN $p \neq 2$, ALL n

When p is odd, we can no longer turn to Pfister forms and their nice properties. However Markus Rost proved a result called Chain Lemma for Milnor K-symbols mod p that is similar to Chain P-equivalence Theorem for Pfister forms that could allow us to keep the same approach as when $p = 2$. We restate it from [4, 5] in its entirety here.

Theorem 5.1. (*Rost's Chain Lemma*) *Let $\{a_1, \dots, a_n\} \in K_n^M(k)/p$ be a nontrivial symbol, where k is a p -special field. Then there exists a smooth projective cellular variety S/k and a collection of invertible sheaves $J = J_1, J'_1, \dots, J_{n-1}, J'_{n-1}$ equipped with nonzero p -forms $\gamma = \gamma_1, \gamma'_1 \dots \gamma_{n-1}, \gamma'_{n-1}$ satisfying the following conditions.*

- (1) $\dim S = p(p^{n-1} - 1) = p^n - p$;
- (2) $\{a_1, \dots, a_n\} = \{a_1, \dots, a_{n-2}, \gamma_{n-1}, \gamma'_{n-1}\} \in K_n^M(k(S))/p$,
 $\{a_1, \dots, a_{i-1}, \gamma_i\} = \{a_1, \dots, a_{i-2}, \gamma_{i-1}, \gamma'_{i-1}\} \in K_i^M(k(S))/p$ for $2 \leq i < n$.
In particular, $\{a_1, \dots, a_n\} = \{\gamma, \gamma'_1, \dots, \gamma'_{n-1}\} \in K_n^M(k(S))/p$;
- (3) $\gamma \notin \Gamma(S, J)^{\otimes(-p)}$, as is evident from (2);
- (4) for any $s \in V(\gamma_i) \cup V(\gamma'_i)$, the field $k(s)$ splits $\{a_1, \dots, a_n\}$;
- (5) $I(V(\gamma_i)) + I(V(\gamma'_i)) \subseteq p\mathbb{Z}$ for all i , as follows from (4);

(6) $\deg(c_1(J)^{\dim S})$ is relatively prime to p .

However, we only care about part (2), which turns one symbol into another symbol through a sequence of equal ones, each time changing just two adjacent spots. The problem is we, or at least I, do not know how to control $\{\gamma, \gamma'_1, \dots, \gamma'_{n-1}\}$. More precisely, given two equal symbols $\{a_1, \dots, a_n\} = \{b_1, \dots, b_n\}$, we do not know if there is a point $s \in S$ such that $\{\gamma(s), \gamma'_1(s), \dots, \gamma'_{n-1}(s)\} = \{b_1, \dots, b_n\}$ in the context of (2). If this is true, we would have the following corollary,

Corollary 5.2. *Given two nontrivial symbols $\{a_1, \dots, a_n\} = \{b_1, \dots, b_n\} \in K_n^M(k)/p$, there exists $a'_1 = a_1, a'_2, \dots, a'_{n-1}, a'_n = b_n$ such that*

- (1) $\{a_1, \dots, a_n\} = s_1 = \dots = s_n = \{b_1, \dots, b_n\}$ where $s_i = \{b_1, \dots, b_{i-1}, a'_i, a_{i+1}, \dots, a_n\}$ for $i = 1, \dots, n$
- (2) more strongly $\{b_1, \dots, b_{i-1}, a'_i, a_{i+1}\} = \{b_1, \dots, b_{i-1}, b_i, a'_{i+1}\}$

For example, when $n = 5$ the changes would be $\{a_1, a_2, a_3, a_4, a_5\} = \{b_1, a'_2, a_3, a_4, a_5\} = \{b_1, b_2, a'_3, a_4, a_5\} = \{b_1, b_2, b_3, a'_4, a_5\} = \{b_1, b_2, b_3, b_4, b_5\}$ while $\{a_1, a_2\} = \{b_1, a'_2\}$, $\{b_1, a'_2, a_3\} = \{b_1, b_2, a'_3\}$, and $\{b_1, b_2, a'_3, a_4\} = \{b_1, b_2, b_3, a'_4\}$ along the way.

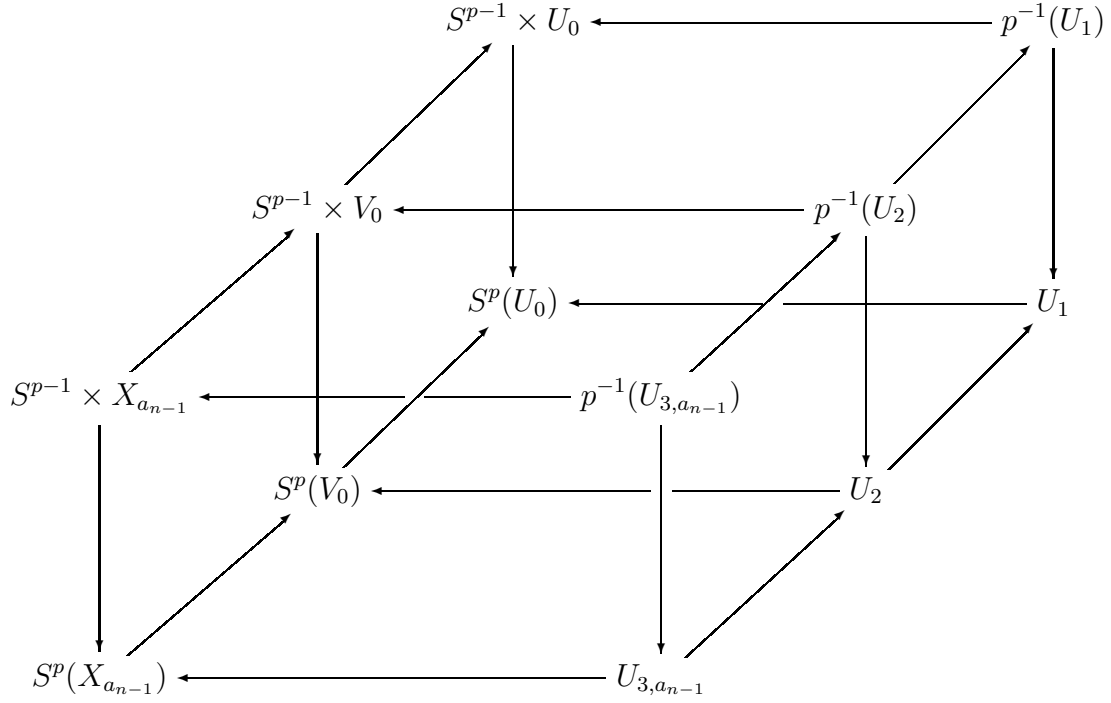
If this corollary holds, we could try to couple it with induction to solve our problem in two steps.

Step 1: write $\{a_1, \dots, a_n\} = \{s_1\} = \dots = \{s_n\} = \{b_1, \dots, b_n\} \in K_n^M(k)/p$ where s_i and s_{i+1} differ in the i^{th} and $(i+1)^{\text{th}}$ spots for all $i = 1, \dots, n-1$. Then we have $X(s_1) \approx X(s_2)$ from the case $n = 2$. By induction we also have $X(s_i) \approx X(s_{i+1})$ for all $2 \leq i \leq n-2$. What remains is to show $X(s_{n-1}) \approx X(s_n)$, (therefore induction is nice to use, but not really needed).

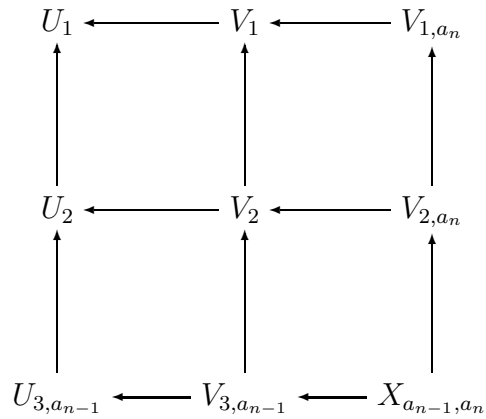
Step 2: show the function field stays the same throughout. One way is to show directly that $X(a_1, \dots, a_{n-2}, a_{n-1}, a_n) \cong X(a_1, \dots, a_{n-2}, a'_{n-1}, a'_n)$ as a_{n-1}, a_n turn into a'_{n-1}, a'_n . For this I have the following idea. Letting $X = X(a_1, \dots, a_{n-2})$, we return to symmetric power construction,

$$\begin{array}{ccccccc}
 S^{p-1}(X) \times X & \longleftarrow & p^{-1}(U_0) & & & & \\
 \downarrow & & \downarrow & & & & \\
 S^p(X) & \longleftarrow & U_0 & \longleftarrow & V_0 & \longleftarrow & X_{a_{n-1}}
 \end{array}$$

Repeating the process for the two rightmost arrows gives



Now taking vector bundles and subvarieties given by $(N - a_n)$ would attach the following diagram to the lower right edge,



Similarly, we have

$$\begin{array}{ccccc}
 U_1 & \longleftarrow & V_1 & \longleftarrow & V_{1,a'_n} \\
 \uparrow & & \uparrow & & \uparrow \\
 U_2 & \longleftarrow & V_2 & \longleftarrow & V_{2,a'_n} \\
 \uparrow & & \uparrow & & \uparrow \\
 U_{3,a'_{n-1}} & \longleftarrow & V_{3,a'_{n-1}} & \longleftarrow & X_{a'_{n-1},a'_n}
 \end{array}$$

Thus we have two intersections X_{a_{n-1},a_n} and $X_{a'_{n-1},a'_n}$ inside the same V_2 . Birationality between them now depends on how the pairs $\{a_{n-1}, a_n\}$ and $\{a'_{n-1}, a'_n\}$ differ in the Chain Lemma, whether

- i) $\{a_{n-1}, a_n\} = \{a'_{n-1}, a'_n\} \in K_2^M(k)/p$ as symbols, or equivalently $(a_{n-1}, a_n, \zeta_p)_k \cong (a'_{n-1}, a'_n, \zeta_p)_k$ as algebras
- ii) $\{a_{n-1}, a_n\} - \{a'_{n-1}, a'_n\} = \sum N()$ some sum of norms.
- iii) or some other way, that may involve a_1, \dots, a_{n-2} .

Another way is to show interchanging two adjacent spots in a symbol does not change its p -generic splitting variety, i.e. $X(a_1, \dots, a_{n-2}, a, b) \approx X(a_1, \dots, a_{n-2}, b, a)$ to move $\{a_{n-1}, a_n\}$ and $\{a'_{n-1}, a'_n\}$ to the first two spots. Then we could consider the Severi-Brauer varieties $SB((a_{n-1}, a_n, \zeta_p)_k)$ and $SB((a'_{n-1}, a'_n, \zeta_p)_k)$ at the start of symmetric power construction. If those starting blocks are the same then the end products would be the same.

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